Torsional oscillations of a plane in a viscous fluid

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On the assumption that the rotational oscillations of a rigid plane are small, boundary-layer type solutions of the Navier–Stokes equations are attempted by an expansion of the velocity components in power series of the amplitude. First- and third-order approximations to the transverse velocity are obtained, from which a correction to the moment on a disk of finite radius is found.

The first non-vanishing approximation to the radial-axial flow (a second-order term) is seen to have a steady component and a component with frequency twice that of the plate. The former component appears to persist outside the boundary layer, and at large distances from the plate to have the character of an irrotational stagnation flow. A re-examination not involving the series approximation reveals that although steady radial flow does exist outside the boundary layer, it is a viscous flow and is confined within a secondary layer. The ratio of the thicknesses of the two layers is found to be inversely proportional to the amplitude of the oscillations. These results indicate that a second-order flow in a region where the first-order flow field vanishes should not be accepted without further discussion.

1. Introduction

The problem of small rotational oscillations of a body of revolution has received considerable attention (see, for example, Kestin & Persen 1954), since it relates to a well-known method of measuring the coefficient of viscosity. Prior to the work of Carrier & di Prima (1956), however, virtually all investigations used Stokes's slow-motion equations, in which the flow was assumed to be entirely circumferential. Consequently, errors arising from the presence of radial flow were neglected.

Carrier & di Prima were principally concerned with evaluating the correction to the torque on a sphere when a second approximation is taken. In the course of the calculations a radial flow was derived but was not discussed in any detail. Commencing with the Navier–Stokes equations in spherical polar co-ordinates, these authors expanded the velocity components in powers of the amplitude of oscillation, and solved the resulting differential equations. The method led to very complicated expressions which were eventually reduced to a practical form by assuming small viscosity.

On the other hand, an early application of the boundary layer equations to time-periodic flows was undertaken by Schlichting (1932), who examined the small two-dimensional oscillations of a cylinder in a stationary fluid. Using a similar small-amplitude expansion, Schlichting found a first-order fluctuating shear layer close to the body. The second approximation, however, yielded a steady secondary flow which did not vanish at large distances from the cylinder and whose magnitude was independent of the viscosity. This effect was ascribed to Reynolds stresses set up in the fluid by the oscillatory motion.

The primary aim of the present paper is to investigate the secondary flows which arise when an infinite plane lamina performs small torsional oscillations in a fluid otherwise at rest. Since for a plane the terms neglected in the boundary-layer approximation vanish identically, the Navier–Stokes equations may be used to derive solutions of the boundary-layer type. If λ is the frequency of the oscillations, then $1/\lambda$ is a typical time, and the boundary-layer thickness will clearly be of order $\sqrt{(\nu/\lambda)}$, where ν is the kinematic viscosity.

Expansion of the velocity components in powers of ϵ, the amplitude of the oscillation, leads to two sets of linear partial differential equations, with simple boundary conditions. The first-order solution is simply the well-known unsteady shear layer for the transverse velocity.

The second-order solution reveals a secondary radial-axial flow composed of a steady term and a term of frequency 2λ. It appears that, as in Schlichting's solution, the steady radial component persists outside the boundary layer. Its form at large distances from the plate is found to be that of an irrotational stagnation flow.

Further considerations result in a correction to the transverse velocity, of third order and containing terms of frequency λ and 3λ. It is found that if $|M|$ is the amplitude of the first-order torque on a disk of radius a, then inclusion of third-order effects leads to a torque of amplitude $|M|\,(1-0{\cdot}101\epsilon^2)$. This contrasts with $|M|\,(1+0{\cdot}015\epsilon^2)$ for the torque on a sphere obtained by Carrier & Di Prima (1956).

A re-examination of the steady radial flow is then undertaken. It is indicated that this flow is set up within the boundary layer by the action of centrifugal force, compared with which the convective inertia terms are neglected in the series approximation. However, outside the boundary layer the centrifugal term becomes vanishingly small, while (axially) inward convection is effective in preventing outward diffusion of vorticity beyond a certain distance. Thus the power series in ϵ ceases to converge outside the boundary layer, and for a correct representation of the flow here the convective terms must be included in the equation of motion.

A solution of the appropriate equation is then obtained, first by an approximate Pohlhausen-type method, and subsequently by numerical means. These solutions show that radial flow, although existing outside the boundary layer, is viscous and is confined within a secondary layer whose thickness is order ϵ^{-1} times that of the shear layer. Outside this secondary layer there is merely a constant axial inflow, as demanded from continuity considerations. Thus we conclude that the irrotational flow, derived from the series approximation and qualitatively similar to that obtained in the two-dimensional case by Schlichting (1932), does not in fact occur.

2. Method of solution

Suppose the plane $z = 0$ to represent a lamina of infinite extent, and the space $z > 0$ to be occupied by an incompressible viscous fluid of density ρ and kinematic viscosity ν. Let (r, ϕ, z) be a set of cylindrical polar co-ordinates, fixed in space, and let the lamina perform torsional oscillations about the axis $r = 0$, the fluid being otherwise at rest. If the amplitude of angular displacement is ϵ, and the frequency of oscillation is λ, then the plate has angular velocity $\omega \cos \lambda t$, or, in complex notation, $\omega e^{i\lambda t}$, where $\omega = \epsilon\lambda$.

Let u, v and w denote respectively the radial, transverse and axial components of velocity, and p the fluid pressure. Then the appropriate Navier–Stokes equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right], \tag{3}$$

while the equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

Since radial symmetry obtains, all derivatives with respect to ϕ are omitted. The relevant boundary conditions of the problem are

$$\left.\begin{aligned} u = 0, \quad v = r\omega e^{i\lambda t}, \quad w = 0 \quad \text{at} \quad z = 0,\\ u \to 0, \quad v \to 0 \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \tag{5}$$

We now attempt a solution of the system (1)–(5) in the form

$$\left.\begin{aligned} u = r\omega F'(\eta,\tau), \quad v = r\omega G(\eta,\tau), \quad w = -2\omega\sqrt{(2\nu/\lambda)}\,F(\eta,\tau),\\ p = p(z,t), \quad z = \sqrt{(2\nu/\lambda)}\,\eta, \quad t = \tau/\lambda, \end{aligned}\right\} \tag{6}$$

where the accent denotes differentiation with respect to η. The continuity equation is satisfied, equations (1) and (2) reduce to the dimensionless forms

$$\frac{\partial F'}{\partial \tau} + \epsilon[F'^2 - G^2 - 2FF''] = \tfrac{1}{2}F''', \tag{7}$$

$$\frac{\partial G}{\partial \tau} + 2\epsilon[F'G - FG'] = \tfrac{1}{2}G'', \tag{8}$$

respectively, and the boundary conditions become

$$F = F' = 0, \quad G = e^{i\tau} \quad \text{at} \quad \eta = 0; \qquad F' \to 0, \quad G \to 0 \quad \text{as} \quad \eta \to \infty. \tag{9}$$

The velocity components u, v and w are now completely determined from (7)–(9), whereupon the pressure can be obtained from (3).

It is next assumed that a solution can be found by expanding F and G in ascending powers of the parameter ϵ. On substituting the series

$$F(\eta,\tau) = F_0(\eta,\tau)+\epsilon F_1(\eta,\tau)+\epsilon^2 F_2(\eta,\tau)+\ldots \tag{10}$$

and

$$G(\eta,\tau) = G_0(\eta,\tau)+\epsilon G_1(\eta,\tau)+\epsilon^2 G_2(\eta,\tau)+\ldots \tag{11}$$

into (7) and (8), and equating coefficients of like powers of ϵ, we obtain the following systems of linear partial differential equations:

$$\frac{\partial F_0'}{\partial \tau} = \tfrac{1}{2}F_0''', \tag{12a}$$

$$\frac{\partial F_1'}{\partial \tau}+F_0'^2-G_0^2-2F_0F_0'' = \tfrac{1}{2}F_1''', \tag{12b}$$

$$\frac{\partial F_2'}{\partial \tau}+2[F_0'F_1'-G_0G_1-F_0F_1''-F_0''F_1] = \tfrac{1}{2}F_2''', \quad \text{etc.}, \tag{12c}$$

$$\frac{\partial G_0}{\partial \tau} = \tfrac{1}{2}G_0'', \tag{13a}$$

$$\frac{\partial G_1}{\partial \tau}+2[F_0'G_0-F_0G_0'] = \tfrac{1}{2}G_1'', \tag{13b}$$

$$\frac{\partial G_2}{\partial \tau}+2[F_0'G_1+F_1'G_0-F_0G_1'-F_1G_0'] = \tfrac{1}{2}G_2'', \quad \text{etc.}, \tag{13c}$$

with boundary conditions

$$F_N = F_N' = 0 \quad \text{at} \quad \eta = 0; \qquad F_N' \to 0 \quad \text{as} \quad \eta \to \infty, \quad N = 0,1,2\ldots, \tag{14}$$

$$G_0 = e^{i\tau}, \quad G_{N+1} = 0 \quad \text{at} \quad \eta = 0; \qquad G_N \to 0 \quad \text{as} \quad \eta \to \infty, \quad N = 0,1,2\ldots. \tag{15}$$

It is clear that the first approximations ($12a$) and ($13a$) are equivalent to neglecting $u(\partial u/\partial r)$, $w(\partial u/\partial z)$, v^2/r compared with $\partial u/\partial t$ in (1), and $u(\partial v/\partial r)$, $w(\partial v/\partial z)$, uv/r compared with $\partial v/\partial t$ in (2). Such an approach was employed by Schlichting (1932), and at this stage it appears to be valid here provided the amplitude ϵ is sufficiently small.

3. First approximation to the transverse velocity

The solution of equation ($12a$) satisfying the boundary conditions (14) is obviously

$$F_0(\eta,\tau) \equiv 0. \tag{16}$$

On the other hand, ($13a$) has the solution

$$G_0(\eta,\tau) = e^{i\tau}\,e^{-(1+i)\eta},$$

which, in real notation, is

$$G_0(\eta,\tau) = e^{-\eta}\cos(\tau-\eta). \tag{17}$$

This is the well-known shear-wave solution for a flat plate oscillating in its own plane in a fluid at rest. Thus its properties have been fully discussed and need not be considered further here. The continuous curves in figure 2 illustrate the function

$$e^{-\eta}\cos(\tau-\eta)-\cos\tau,$$

which is merely $G_0(\eta,\tau)$ taken relative to the oscillating plate.

Since $F_0 = 0$, the first-order solution of the system (1)–(5) is a transverse velocity given by

$$v = r\omega\, e^{-\sqrt{(\lambda/2\nu)}z}\cos(\lambda t-\sqrt{(\lambda/2\nu)}\,z), \tag{18}$$

and no radial or axial velocity. The transverse shearing stress is defined to be

$$\tau_t = \rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0}, \tag{19}$$

so that the first approximation is, from (18),

$$\tau_t = -\rho r\omega\sqrt{(\nu\lambda)}\cos(\lambda t + \tfrac{1}{4}\pi). \tag{20}$$

This shearing stress is seen to have a phase lead of $\frac{1}{4}\pi$ over the oscillations of the plate.

Although we are dealing with a plane of infinite extent, the results obtained above may be applied to a circular disk of radius a provided edge effects can be neglected. This seems to be justified if a is large compared with the thickness of the boundary layer. In this case the frictional torque (for the two sides of the disk) is

$$M = -4\pi\int_0^a r^2\tau_t\,dr. \tag{21}$$

Hence, using (20), the first approximation to the moment is

$$M = \pi a^4\rho\epsilon\sqrt{(\nu\lambda^3)}\cos(\lambda t + \tfrac{1}{4}\pi). \tag{22}$$

These results are well known.

4. Radial and axial velocities

Since $F_0 = 0$, the first non-vanishing approximations to the radial and axial velocities are obtainable from the second-order solution F_1 of equation (12*b*). Substituting from (17) into (12*b*), we have

$$\frac{\partial F_1'}{\partial\tau} - \tfrac{1}{2}e^{-2\eta}[1 + e^{2i\tau}e^{-2i\eta}] = \tfrac{1}{2}F_1''', \tag{23}$$

with boundary conditions

$$F_1(0) = 0, \quad F_1'(0) = 0, \quad F_1'(\infty) = 0. \tag{24}$$

The form of (23) invites a substitution

$$F_1(\eta,\tau) = f(\eta) + h(\eta)\,e^{2i\tau}, \tag{25}$$

and this leads to the two equations

$$2ih' - \tfrac{1}{2}e^{-2(1+i)\eta} = \tfrac{1}{2}h''', \tag{26a}$$

$$-e^{-2\eta} = \tfrac{1}{2}f''', \tag{26b}$$

with $\quad f(0) = f'(0) = h(0) = h'(0) = 0; \quad f'(\infty) = h'(\infty) = 0.$

The solution of (26*a*) is readily seen to be

$$h = A + B\,e^{-\sqrt{2(1+i)}\,\eta} + C\,e^{\sqrt{2(1+i)}\,\eta} - \frac{1+i}{16}e^{-2(1+i)\eta},$$

where A, B and C are integration constants. Application of the boundary conditions immediately leads to

$$A = \frac{1+i}{16}(1-\sqrt{2}), \quad B = \frac{1+i}{8\sqrt{2}}, \quad C = 0;$$

whereupon we have

$$h = \frac{1+i}{16}[1-\sqrt{2}+\sqrt{2}\,e^{-\sqrt{2(1+i)}\eta}-e^{-2(1+i)\eta}], \tag{27a}$$

and

$$h' = -\frac{i}{4}[e^{-\sqrt{2(1+i)}\eta}-e^{-2(1+i)\eta}]. \tag{27b}$$

Equation (26*b*) has the solution

$$f = A+B\eta+C\eta^2+\tfrac{1}{8}e^{-2\eta}.$$

The requirements $f(0) = 0$ and $f'(0) = 0$ yield $A = -\frac{1}{8}$, $B = \frac{1}{4}$, while in order that f' should remain finite as $\eta \to \infty$ it is necessary that $C = 0$. These values give

$$f = -\tfrac{1}{8}[1-2\eta-e^{-2\eta}], \tag{28a}$$

and

$$f' = \tfrac{1}{4}[1-e^{-2\eta}], \tag{28b}$$

which, however, does not satisfy $f'(\eta) \to 0$ as $\eta \to \infty$. The significance of this result will be discussed subsequently.

Reverting to real notation, we have from (25), (27) and (28)

$$F_1(\eta,\tau) = -\tfrac{1}{8}(1-2\eta-e^{-2\eta})-\tfrac{1}{16}\{(2-\sqrt{2})\cos(2\tau+\tfrac{1}{4}\pi) \\ -2\,e^{-\sqrt{2}\eta}\cos(2\tau-\sqrt{2}\eta+\tfrac{1}{4}\pi)+\sqrt{2}\,e^{-2\eta}\cos(2\tau-2\eta+\tfrac{1}{4}\pi)\}, \tag{29}$$

and

$$F_1'(\eta,\tau) = \tfrac{1}{4}(1-e^{-2\eta})+\tfrac{1}{4}\{e^{-\sqrt{2}\eta}\sin(2\tau-\sqrt{2}\eta)-e^{-2\eta}\sin(2\tau-2\eta)\}. \tag{30}$$

Equations (29) and (30) show that the radial and axial velocities are each composed of a steady term and an unsteady term of frequency twice that of the torsional oscillations. Indeed, this was to be anticipated on physical grounds. For a fluid particle adjacent to the wall and distance r from the axis experiences a centrifugal force $\rho r\Omega^2$ per unit volume, where Ω is the angular velocity. Since $\Omega = \omega\cos\lambda t$, the centrifugal force is $\rho r\omega^2\cos^2\lambda t = \frac{1}{2}\rho r\omega^2(1+\cos 2\lambda t)$, having the two components mentioned.

We may therefore divide the radial and axial velocities u and w each into a steady part, denoted respectively by u_s and w_s, and a fluctuating part u_f, w_f. From equations (6), (29) and (30), these are given by

$$u_s = \frac{r\omega\epsilon}{4}\{1-e^{-\sqrt{(2\lambda/\nu)}z}\}, \tag{31}$$

$$w_s = \frac{\omega\epsilon}{4}\sqrt{\left(\frac{2\nu}{\lambda}\right)}\{1-\sqrt{(2\lambda/\nu)}\,z-e^{-\sqrt{(2\lambda/\nu)}z}\}, \tag{32}$$

$$u_f = \frac{r\omega\epsilon}{4}\{e^{-\sqrt{(\lambda/\nu)}z}\sin(2\tau-\sqrt{(\lambda/\nu)}\,z)-e^{-\sqrt{(2\lambda/\nu)}z}\sin(2\tau-\sqrt{(2\lambda/\nu)}\,z)\}, \tag{33}$$

$$w_f = \frac{\omega\epsilon}{8}\sqrt{\left(\frac{2\nu}{\lambda}\right)}\{(2-\sqrt{2})\cos(2\tau+\tfrac{1}{4}\pi)-2\,e^{-\sqrt{(\lambda/\nu)}z}\cos(2\tau-\sqrt{(\lambda/\nu)}\,z+\tfrac{1}{4}\pi) \\ +\sqrt{2}\,e^{-\sqrt{(2\lambda/\nu)}z}\cos(2\tau-\sqrt{(2\lambda/\nu)}\,z+\tfrac{1}{4}\pi)\}. \tag{34}$$

Relative to the oscillations of the plate, the unsteady radial velocity u_f is seen to have a phase lag which tends to $\frac{1}{2}\pi$ at the plate. Moreover u_f decreases exponentially with distance from the wall, the rate of diminution depending on the factor $\sqrt{(\nu/\lambda)}$, which is the order of thickess of the boundary layer. That is, the fluctuating radial component becomes negligible outside the boundary layer.

This is shown in figure 1, which illustrates the variation of the non-dimensionalized velocity $4u_f/r\omega\epsilon$ with $\eta = \sqrt{(\lambda/2\nu)}\,z$ at selected times.

The unsteady axial term w_f has a phase lead of $\frac{1}{4}\pi$ over the transverse velocity. Outside the boundary layer, continuity demands that some unsteady axial velocity should persist, and in fact, for large z,

$$w_f \sim \frac{\omega\epsilon}{8}\sqrt{\left(\frac{2\nu}{\lambda}\right)}\,(2-\sqrt{2})\cos(2\tau+\tfrac{1}{4}\pi),$$

which is quite small for small viscosity.

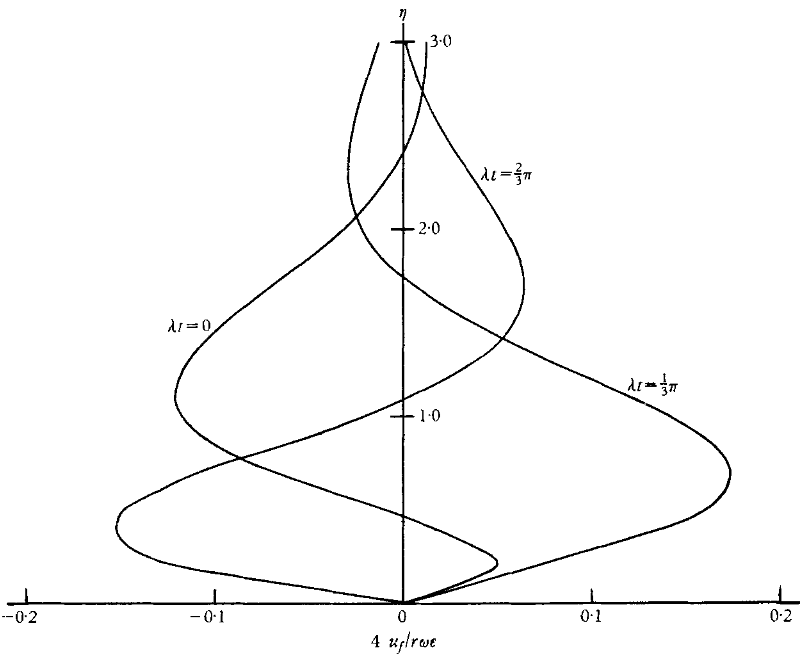

FIGURE 1. Variation of $4u_f/r\omega\epsilon$ with $\eta = (\lambda/2\nu)\,z$ at times (i) $\lambda t = 0$, (ii) $\lambda t = \frac{1}{3}\pi$, (iii) $\lambda t = \frac{2}{3}\pi$.

The centrifugal and shearing forces at the plate give rise to steady components of radial and axial velocity. The radial velocity u_s, as seen from equation (31), consists of two terms, one of which decreases exponentially with z, and so vanishes outside the boundary layer, whereas the other is independent of distance from the plate. Thus for large z,

$$u_s \sim \frac{r\omega\epsilon}{4}. \tag{35}$$

Consequently, it would appear that, unlike the fluctuating component, a steady radial flow persists outside the boundary layer. Furthermore, this flow has the important feature that its magnitude is independent of the value of the kinematic viscosity.

Correspondingly, the asymptotic axial flow outside the boundary layer is, from (32),

$$w_s \sim \frac{\omega\epsilon}{4}\sqrt{\left(\frac{2\nu}{\lambda}\right)}\left\{1-\sqrt{\left(\frac{2\lambda}{\nu}\right)}\,z\right\}. \tag{36}$$

Here the first term represents the axial velocity required by continuity to balance the radial flow within the boundary layer, while the second term, independent of viscosity, balances the asymptotic radial flow outside the boundary layer.

The form of equations (31) and (32) permits the definition of a stream function for the steady radial-axial flow, namely

$$\psi = \frac{r^2\omega\epsilon}{4}\sqrt{\left(\frac{\nu}{2\lambda}\right)}\left[\sqrt{\left(\frac{2\lambda}{\nu}\right)}z - 1 + e^{-\sqrt{(2\lambda/\nu)}z}\right]$$

such that

$$u_s = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad w_s = -\frac{1}{r}\frac{\partial\psi}{\partial r},$$

satisfying continuity requirements. At large distances from the plate we have that

$$\psi \sim \frac{r^2\omega\epsilon}{4}\left\{z - \sqrt{\left(\frac{\nu}{2\lambda}\right)}\right\}, \tag{37}$$

which is the stream function of an axially symmetrical stagnation flow against an imaginary wall distant $z = \sqrt{(\nu/2\lambda)}$ from the plate. Thus we have achieved the apparent result that torsional oscillations of the plate induce a steady potential flow which is such that the smaller the viscosity the closer its streamlines approach the plate. We re-examine this result in § 6.

As a matter of interest we calculate an approximate expression for the pressure gradient normal to the disk. From equations (3) and (6), we find

$$-\frac{1}{\rho\omega\lambda}\sqrt{\left(\frac{\lambda}{2\nu}\right)}\frac{\partial p}{\partial z} = -2\frac{\partial F}{\partial\tau} + 4\epsilon FF' + F''$$

$$\doteqdot \epsilon\left[F_1'' - 2\frac{\partial F_1'}{\partial\tau}\right],$$

to first order in ϵ. Substituting from (29) and (30), we obtain eventually

$$-\frac{1}{\rho\omega\lambda}\sqrt{\left(\frac{\lambda}{2\nu}\right)}\frac{\partial p}{\partial z} = \frac{\epsilon}{4}[2\,e^{-\sqrt{(2\lambda/\nu)}z} + \sqrt{2}\,e^{-\sqrt{(2\lambda/\nu)}z}\sin(2\lambda t - \sqrt{(2\lambda/\nu)}\,z + \tfrac{1}{4}\pi)$$
$$- (2-\sqrt{2})\sin(2\lambda t + \tfrac{1}{4}\pi)]. \tag{38}$$

The contribution from the potential-type flow is of order ϵ^2, and so has been neglected here.

Finally, we calculate the radial shearing stress at the plate,

$$\tau_r = \rho\nu\left(\frac{\partial u}{\partial z}\right)_{z=0}.$$

Substituting for u from (31) and (33), we find

$$\tau_r = \frac{\rho\nu r\omega\epsilon}{4}\sqrt{\left(\frac{2\lambda}{\nu}\right)}[1 + (\sqrt{2}-1)\sin(2\lambda t + \tfrac{1}{4}\pi)]. \tag{39}$$

5. Third approximation to the transverse velocity

Since $F_0 = 0$, equation (13*b*) becomes

$$\frac{\partial G_1}{\partial\tau} = \tfrac{1}{2}G_1'',$$

of which the solution satisfying $G_1(0) = 0$, $G_1(\infty) = 0$ is clearly

$$G_1(\eta,\tau) = 0.$$

(It can moreover easily be demonstrated that

$$F_{2N}(\eta,\tau) = 0, \quad G_{2N+1}(\eta,\tau) = 0, \quad N = 0,1,2\ldots .)$$

Thus the second approximation to the transverse velocity is zero, and the third approximation is given from $G_2(\eta,\tau)$, which satisfies the equation

$$\frac{\partial G_2}{\partial \tau} + 2[F_1' G_0 - F_1 G_0'] = \tfrac{1}{2} G_2'', \tag{40}$$

with boundary conditions $G_2(0) = 0$, $G_2 \to 0$ as $\eta \to \infty$. From equations (17), (29) and (30) we find that, in complex notation,

$$\begin{aligned} F_1' G_0 - F_1 G_0' = \tfrac{1}{4} e^{i\tau} \Big\{ & \tfrac{1}{2}(1-i)\, e^{-(1+i)\eta} + (1+i)\,\eta\, e^{-(1+i)\eta} - (\tfrac{3}{4} - i)\, e^{-(3+i)\eta} \\ & - \tfrac{1}{4}(\sqrt{2}-1)\, e^{-(1-i)\eta} + \left(\frac{\sqrt{2}}{4} - \frac{i}{2}\right) e^{-[\sqrt{2}+1+i(\sqrt{2}-1)]\eta} \Big\} \\ & + \frac{i}{16} e^{3i\tau} \{ e^{-3(1+i)\eta} - (2-\sqrt{2})\, e^{-(\sqrt{2}+1)(1+i)\eta} - (\sqrt{2}-1)\, e^{-(1+i)\eta} \}. \end{aligned} \tag{41}$$

The form of (41) suggests that a solution of equation (40) could be

$$G_2(\eta,\tau) = \chi(\eta)\, e^{i\tau} + \zeta(\eta)\, e^{3i\tau}, \tag{42}$$

and on substituting (41) and (42) into (40), we obtain the pair of linear differential equations

$$\begin{aligned} \chi'' - 2i\chi = {} & \tfrac{1}{2}(1-i)\, e^{-(1+i)\eta} + (1+i)\,\eta\, e^{-(1+i)\eta} - (\tfrac{3}{4} - i)\, e^{-(3+i)\eta} \\ & - \tfrac{1}{4}(\sqrt{2}-1)\, e^{-(1-i)\eta} + \left(\frac{\sqrt{2}}{4} - \frac{i}{2}\right) e^{-[\sqrt{2}+1+i(\sqrt{2}-1)]\eta}, \end{aligned} \tag{43a}$$

$$\zeta'' - 6i\zeta = \frac{i}{4} [e^{-3(1+i)\eta} - (2-\sqrt{2})\, e^{-(\sqrt{2}+1)(1+i)\eta} - (\sqrt{2}-1)\, e^{-(1+i)\eta}], \tag{43b}$$

with boundary conditions

$$\chi(0) = \zeta(0) = \chi(\infty) = \zeta(\infty) = 0.$$

The real and imaginary parts of the solutions χ, ζ of (43 *a*) and (43 *b*) are found to be

$$\begin{aligned} \mathscr{R}\{\chi\} = {} & -\cos\eta\{[\tfrac{3}{80} + \tfrac{1}{8}\eta + \tfrac{1}{4}\eta^2]\, e^{-\eta} + \tfrac{1}{40} e^{-3\eta}\} + \sin\eta\left\{\left[\frac{15\sqrt{2}-21}{80} + \tfrac{3}{8}\eta\right] e^{-\eta} + \tfrac{11}{80} e^{-3\eta}\right\} \\ & + \tfrac{1}{16} e^{-(\sqrt{2}+1)\eta} [\cos(\sqrt{2}-1)\,\eta - \sqrt{2}\sin(\sqrt{2}-1)\,\eta], \end{aligned} \tag{44a}$$

$$\begin{aligned} \mathscr{I}\{\chi\} = {} & \cos\eta\left\{\left[\frac{5\sqrt{2}-11}{80} + \tfrac{3}{8}\eta\right] e^{-\eta} + \tfrac{11}{80} e^{-3\eta}\right\} + \sin\eta\{[\tfrac{3}{80} + \tfrac{1}{8}\eta + \tfrac{1}{4}\eta^2]\, e^{-\eta} + \tfrac{1}{40} e^{-3\eta}\} \\ & - \tfrac{1}{16} e^{-(\sqrt{2}+1)\eta} [\sqrt{2}\cos(\sqrt{2}-1)\,\eta + \sin(\sqrt{2}-1)\,\eta], \end{aligned} \tag{44b}$$

$$\begin{aligned} \mathscr{R}\{\zeta\} = {} & \tfrac{1}{48}\{-e^{-\sqrt{3}\eta}\cos\sqrt{3}\eta + e^{-3\eta}\cos 3\eta - 3(\sqrt{2}-1) \\ & \times [e^{-(\sqrt{2}+1)\eta}\cos(\sqrt{2}+1)\,\eta - e^{-\eta}\cos\eta]\}, \end{aligned} \tag{44c}$$

$$\begin{aligned} \mathscr{I}\{\zeta\} = {} & \tfrac{1}{48}\{e^{-\sqrt{3}\eta}\sin\sqrt{3}\eta - e^{-3\eta}\sin 3\eta + 3(\sqrt{2}-1) \\ & \times [e^{-(\sqrt{2}+1)\eta}\sin(\sqrt{2}+1)\,\eta - e^{-\eta}\sin\eta]\}. \end{aligned} \tag{44d}$$

From equation (42) we have that, in real notation,

$$G_2 = \mathscr{R}\{\chi\}\cos\tau - \mathscr{I}\{\chi\}\sin\tau + \mathscr{R}\{\zeta\}\cos 3\tau - \mathscr{I}\{\zeta\}\sin 3\tau, \tag{45}$$

so that to third order the transverse velocity is

$$v = r\omega(G_0 + \epsilon^2 G_2), \tag{46}$$

with G_0 and G_2 given respectively by (17) and (46). Figure 2 illustrates the dimensionless transverse velocity relative to a set of axes fixed in the plate. The continuous curves represent, for selected times, the function $G_0 - \cos\tau$, while the broken curves represent

$$G_0 + \epsilon^2 G_2 - \cos\tau$$

for the case $\epsilon = \frac{1}{2}$. The vertical straight lines are the values of $\cos\tau$ at the relevant times. It is seen that the difference between the first and third approximations is quite small close to the plate and rather larger at moderate distances (inside the boundary layer).

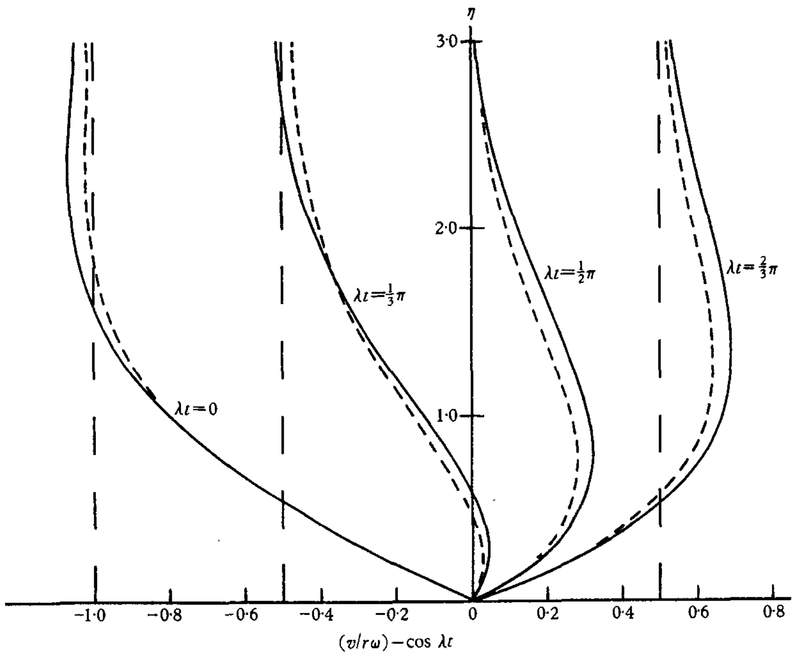

FIGURE 2. Variation of $(v/r\omega) - \cos\lambda t$ with $\eta = \sqrt{(\lambda/2\nu)}\, z$ at times (i) $\lambda t = 0$, (ii) $\lambda t = \frac{1}{3}\pi$, (iii) $\lambda t = \frac{1}{2}\pi$, (iv) $\lambda t = \frac{2}{3}\pi$: ——— first approximation; — — — — second approximation.

Using (44)–(46), we now find for the fluctuating shearing stress at the wall

$$\tau_t = \rho\nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \rho\nu r\omega\sqrt{\left(\frac{\lambda}{2\nu}\right)}$$
$$\times\{\sin\tau(1 - 0{\cdot}262\epsilon^2) - 0{\cdot}012\epsilon^2\sin 3\tau - \cos\tau(1 + 0{\cdot}060\epsilon^2) + 0{\cdot}012\epsilon^2\cos 3\tau\}. \tag{47}$$

Hence by (21) the moment on a disk of radius a is, to a third approximation,

$$M' = \frac{\pi a^4 \rho\epsilon}{\sqrt{2}}\sqrt{(\nu\lambda^3)}$$
$$\times\{\cos\tau(1 + 0{\cdot}060\epsilon^2) - 0{\cdot}012\epsilon^2\cos 3\tau - \sin\tau(1 - 0{\cdot}262\epsilon^2) + 0{\cdot}012\epsilon^2\sin 3\tau\}. \tag{48}$$

From this equation and (22), we find for the magnitude of the fluctuating torque

$$|M'| = |M|\{1-0{\cdot}101\epsilon^2\}. \tag{49}$$

As a matter of interest it can easily be verified that the result (49) remains unaltered if the terms in $\cos 3\tau$, $\sin 3\tau$ are omitted from the preceding formulae. This means that, to third order, the third harmonic terms only have an effect on the phase of the fluctuating torque.

We see that the third approximation to the transverse velocity necessitates a negative correction to the moment of $0{\cdot}101\epsilon^2|M|$; that is, a decrease of about 2·5 % when $\epsilon = \frac{1}{2}$ and 0·6 % when $\epsilon = \frac{1}{4}$. These values may be contrasted with those obtained, by a somewhat different procedure, by Carrier & Di Prima (1956) for a torsionally oscillating sphere. Their result, translated into our notation, was

$$|M'| = |M|\{1+0{\cdot}015\epsilon^2\},$$

yielding an increase of about 0·4 % when $\epsilon = \frac{1}{2}$ and 0·1 % when $\epsilon = \frac{1}{4}$. Thus it would seem that for a disk the increment to the torque is about 7 times as great as for a sphere and of opposite sign.

Finally, it should be noted that the results (47)–(49) have value only if equations (29) and (30) constitute a good approximation to the second-order radial-axial flow close to the plate. It is indicated in § 6 that this is the case, even though they are a bad approximation at larger distances.

6. The steady radial flow

At this stage a discussion of the results obtained so far is worth-while. As demonstrated in figure 2, the value of the transverse velocity is not greatly altered when the third approximation is taken in place of the first; and, as expected, the third approximation vanishes outside the boundary layer, even though its evaluation involves the non-vanishing radial-axial terms. Hence, there is no reason to doubt the validity of the series expansion (11), and therefore we assume in the subsequent calculations that the transverse velocity is given, to a good approximation, by

$$v = r\omega\, e^{-\sqrt{(\lambda/2\nu)}z} \cos\{\lambda t - \sqrt{(\lambda/2\nu)}\, z\}. \tag{18 bis}$$

Moreover, we assume that equations (33) and (34) adequately represent the unsteady radial and axial velocities, and do not consider them further here. To establish this would of course necessitate quite complicated calculations.

On the other hand, the unexpected form of the steady radial-axial component suggests that in this case a further investigation is warranted. Assuming that u/r is independent of r, and that there is no radial pressure gradient, the steady radial and axial velocities may be taken to be governed by the equations

$$u_s\frac{\partial u_s}{\partial r} + w_s\frac{\partial u_s}{\partial z} - \frac{\bar{v}^2}{r} = \nu\frac{\partial^2 u_s}{\partial z^2} \tag{50}$$

and

$$\frac{\partial u_s}{\partial r} + \frac{u_s}{r} + \frac{\partial w_s}{\partial z} = 0, \tag{51}$$

where $\bar{v}$ is the root-mean-square value of the transverse velocity (18); that is,

$$\bar{v} = \frac{r\omega}{\sqrt{2}} e^{-\sqrt{(\lambda/2\nu)}\,z}. \tag{52}$$

The boundary conditions to be satisfied are

$$u_s = w_s = 0 \quad \text{at} \quad z = 0; \qquad u_s \to 0 \quad \text{as} \quad z \to \infty. \tag{53}$$

On the assumption that the series (10) for u_s, w_s is valid, the first approximation to u_s is given from the equation

$$\nu \frac{\partial^2 u_s}{\partial z^2} = -\frac{\bar{v}^2}{r}, \tag{54}$$

whose solution is (31). At large distances from the plate $\bar{v}$ is vanishingly small and consequently the asymptotic equation for u_s is

$$\nu \frac{\partial^2 u_s}{\partial z^2} = 0. \tag{55}$$

This explains why the radial flow at large distances is independent of viscosity; and the solution of (55) is equation (35).

This solution, however, is acceptable only if the convective inertia terms $u_s(\partial u_s/\partial r)$, $w_s(\partial u_s/\partial z)$ can be neglected for all z, and it is clear on physical grounds that this cannot be the case. The vorticity created at the plate has imparted to it a radial velocity component as a result of centrifugal action. This vorticity diffuses away from the plate and beyond the boundary layer. But, as the edge of the layer is approached, the transverse velocity becomes vanishingly small, so that radial motion is no longer maintained by centrifugal force and can persist only because of the diffusion of vorticity to large distances. At the same time the convection of the fluid by itself, hitherto negligible, will be principally axial—towards the plate—as it has to balance the fluid forced radially out by centrifugal action. It must therefore be anticipated that instead of persisting at large z, diffusion of vorticity, and consequently steady radial flow, will be limited to within a finite distance of the plate—a distance determined by the counteraction of outward diffusion and inward convection.

Applying these considerations to the equation of motion (50) we see that the inertia terms $u_s(\partial u_s/\partial r)$, $w_s(\partial u_s/\partial z)$ are negligible compared with the centrifugal term $\bar{v}^2/r$ only in the region where the latter is non-vanishing, that is, within the boundary layer. Elsewhere the inertia forces are at least comparable with the centrifugal force, and so outside the boundary layer a correct estimation of the flow must take account of them. Alternatively, this means that the series expansion (10) in powers of ϵ yields valid approximations inside the boundary layer only.

(An interesting parallel to these results is afforded by the well-known Oseen's equations for flow past an obstacle at low Reynolds numbers. In this case it appears that close to the obstacle inertia forces are negligible and Stokes' equations provide a good approximation. But at large distances these inertia forces, though still very small, are now of similar magnitude to the viscous forces. Thus Stokes' equations are no longer satisfactory and Oseen's equations are required for a reasonable approximation to the flow field.)

We now seek a solution of equations (50)–(53) which does not involve the approximation arising from the series (10). On writing

$$u_s = r\omega\epsilon f'(\eta), \quad w_s = -2\omega\epsilon\sqrt{\left(\frac{2\nu}{\lambda}\right)}f(\eta), \quad z = \sqrt{\left(\frac{2\nu}{\lambda}\right)}\eta, \tag{56}$$

equation (50) becomes

$$\epsilon^2(f'^2 - 2ff'') - \tfrac{1}{2}e^{-2\eta} = \tfrac{1}{2}f''' \tag{57}$$

The steady radial flow will, by the foregoing argument, be confined within some secondary layer whose thickness, however, may be large compared with that of the primary layer discussed so far. By use of a Pohlhausen-type method, an approximate solution may be obtained which gives an estimate of the thickness of this secondary layer.

We suppose that the flow takes place within a distance d of the plate. Then we seek a solution of the integral of equation (57) over the range 0 to d, with the assumption that conditions at ∞ are satisfied at $\eta = d$. The relevant boundary conditions will then be

$$\text{at } \eta = 0\colon \quad f = 0, \quad f' = 0, \quad f''' = -1,$$
$$\text{at } \eta = d\colon \quad f' = f'' = f''' = \ldots = 0,$$

while integration of (57) now gives

$$3\epsilon^2\int_0^d f'^2\,d\eta - \tfrac{1}{4} = -\tfrac{1}{2}f''(0). \tag{58}$$

A solution of (58) may take the form

$$f' = -\tfrac{1}{4}\{e^{-2\eta} - e^{-\eta/d}\}. \tag{59}$$

This gives $f' = 0$ at $\eta = 0$; moreover, assuming that $d \gg 1$, that is, that the secondary layer is considerably thicker than the primary layer, we have

$$f'''(0) = -1 + O(d^{-2}),$$

which nearly satisfies the condition on $f'''(0)$. When $\eta \ll 1$, $e^{-\eta/d} \doteqdot 1$, so that (59) behaves like the function f' of equation (28*b*) close to the plate.

Substitution from (59) into (58) leads to

$$\frac{3\epsilon^2}{2}\left\{\frac{1}{4} - \frac{2d}{2d+1} + \frac{d}{2}\right\} = \frac{1}{d} \tag{60}$$

as the equation determining d. It may be noticed that if the terms on the left-hand side of (60), which represent the convection, are neglected, we obtain $d = \infty$, as expected. As it stands, however, equation (60) yields

$$d \doteqdot \frac{1\cdot155}{\epsilon} + 0\cdot250 \tag{61}$$

for the order of thickness of the secondary layer. Equation (61) shows that if d_1 is the thickness when $\epsilon = \frac{1}{4}$, and d_2 that when $\epsilon = \frac{1}{2}$, then

$$\frac{d_1}{d_2} \doteqdot 1\cdot90. \tag{62}$$

Thus we have that provided ϵ is sufficiently small, steady radial flow takes place within a layer of thickness $O(\epsilon^{-1})$ times that of the primary layer.

In other words, the order of thickness of the secondary layer is

$$\epsilon^{-1}\surd(\nu/\lambda) = \surd(\nu/\omega\epsilon),$$

and this could have been estimated from first principles. For steady flow occurs in a region where inertia forces are comparable in magnitude with viscous forces, that is, where $u_s(\partial u_s/\partial r) \sim \nu(\partial^2 u_s/\partial z^2)$. Since $u_s \sim r\omega\epsilon$, and taking $z \sim \delta$, the layer thickness, we find $r\omega^2\epsilon^2 \sim \nu(r\omega\epsilon/\delta^2)$, which gives $\delta \sim \surd(\nu/\omega\epsilon)$.

Outside the secondary layer there is merely a constant axial inflow whose value is obtained from integration of (59). We find that

$$f(\infty) = \tfrac{1}{4}(d-\tfrac{1}{2}),$$

which leads to

$$f \doteqdot 1{\cdot}09 \quad \text{when} \quad \epsilon = \tfrac{1}{4}, \qquad f \doteqdot 0{\cdot}52 \quad \text{when} \quad \epsilon \doteqdot \tfrac{1}{2}. \tag{63}$$

From equation (56) we now have

$$w_s(\infty) = O(\epsilon\surd(\nu\lambda)),$$

technically a first-order effect. But this is explained by the continuity equation which requires that $\partial w_s/\partial z \sim u_s/r$, i.e. that $w_s \sim \delta,\ \omega\epsilon \sim \epsilon\surd(\nu\lambda)$.

Of course, equation (59) is no more than a rough approximation to the solution of (57), useful in obtaining an estimate of d. In order to deduce a correct representation of the functions f and f', it is necessary to resort to numerical solutions of equation (57). Such solutions were calculated with the aid of the Mercury Automatic Computer at the University of Manchester, for the two values $\epsilon = \frac{1}{4}$ and $\epsilon = \frac{1}{2}$.

These solutions yielded the following results. Taking $f' \leqslant 0{\cdot}01$ as a measure of the layer within which radial flow is confined, it was found that

$$\text{for } \epsilon = \tfrac{1}{4}, \quad f' = 0{\cdot}01 \quad \text{when} \quad \eta = d_1 = 13{\cdot}16;$$
$$\text{for } \epsilon = \tfrac{1}{2}, \quad f' = 0{\cdot}01 \quad \text{when} \quad \eta = d_2 = 7{\cdot}24.$$

These give

$$\frac{d_1}{d_2} \doteqdot 1{\cdot}82,$$

in close agreement with (62). This tends to confirm our conclusion that steady flow occurs within a layer whose order of thickness is $\epsilon^{-1}\surd(2\nu/\lambda)$. Again, it was found that for large η, f tends asymptotically to $0{\cdot}99$ when $\epsilon = \frac{1}{4}$, and to $0{\cdot}50$ when $\epsilon = \frac{1}{2}$. These values for the inflow also agree with the approximate values (63).

The steady components of radial and axial velocity are illustrated in figures 3 and 4, respectively. In figure 3, the curve designated (I) represents the solution f' given by equation (28b) which is independent of ϵ and becomes irrotational for large η. The curves (II) and (III) are the function f' obtained from the numerical solution of (57) for $\epsilon = \frac{1}{4}$ and $\epsilon = \frac{1}{2}$, respectively. A similar notation is used for the curves of f in figure 4.

It is immediately clear that well within the primary layer, say for $\eta < \frac{1}{2}$, the values of f' and f derived from the power series expansion are very close to those obtained from the numerical solution. This means that, as indicated earlier, very close to the plate the dominant factor in determining the radial flow is the centrifugal force, and neglect of convection is justified.

On the other hand, for $\eta > \frac{1}{2}$ the curves begin to diverge, taking the forms illustrated. The radial velocity given by the numerical solution tends to zero instead of a finite value, thus satisfying the required boundary condition, at infinity. Similarly, the axial velocity tends asymptotically to a constant inflow in place of a linear function of distance from the plate. The divergence between the potential-flow solution and the secondary-layer solution decreases with ϵ, which is to be anticipated from an inspection of (57).

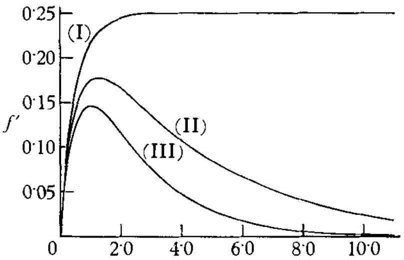

FIGURE 3

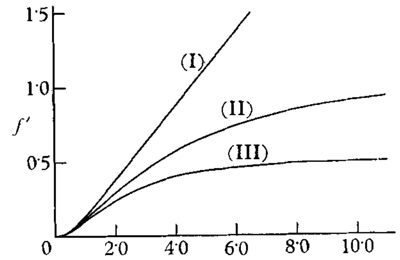

FIGURE 4

FIGURE 3. The steady radial flow. Variation of $f' = u_s/r\omega\epsilon$ with $\eta = \sqrt{(\lambda/2\nu)}z$. (I) f' from series expansion; (II) f' from numerical solution, $\epsilon = \frac{1}{4}$; (III) f' from numerical solution, $\epsilon = \frac{1}{2}$.

FIGURE 4. The steady axial flow. Variation of $f = -w_s/2\omega\epsilon\sqrt{(\lambda/2\nu)}$ with $\eta = \sqrt{(\lambda/2\nu)}z$. (I) f from series expansion; (II) f from numerical solution, $\epsilon = \frac{1}{4}$; (III) f from numerical solution, $\epsilon = \frac{1}{2}$.

Thus we conclude that a true representation of the induced steady radial-axial flow can only be obtained by including the convective inertia terms. These terms, although small, play a decisive role in determining the character of the flow at large distances from the plate.

It is noteworthy that our solution is consistent with the well-known Karman rotating-disk solution for steady flow. In the latter it is found that radial flow is confined within the boundary layer of the transverse velocity. Here also, inward axial flow prevents outward diffusion of vorticity beyond the layer, whose thickness is of order $\sqrt{(\nu/\omega)}$, ω being the steady angular velocity of the disk.

In conclusion it must be remarked that, with the new values of f' and f, the third approximation to the transverse velocity will be altered. But since there has been little variation in the values of f' and f close to the plate, it is unlikely that the transverse velocity will suffer much change except possibly towards the edge of the layer, where it in any case tends to zero. Consequently, we expect the third-order values of shearing stress and moment derived in § 5 to be valid.

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